Numerical Methods, October 23, 2018

Solving sets of linear equations.

Recall vectors, matrices, summation and multiplication. Compute

$$\begin{pmatrix} 1 & 0 & 5 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \\ -2 \end{pmatrix}.$$

Matrix A of the system and the vector of the right hand side b are

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}.$$

Linear equations with unknowns $x_1, x_2, \ldots, x_n$ are

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ & & & & & & & \ldots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m. \end{array}$$

Matrix A of the system and the vector on the right hand side b are

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We search for x fulfilling $Ax = b$ (matrix-vector multiplication).

If $n \neq m$ we never obtain a unique solution $\Rightarrow$

We will consider only $n = m$ and assume that there exists a unique solution of $Ax = b$.

Gaussian algorithm.

Gaussian elementary row operations to transform the augmented matrix $(A|b)$ to the echelon form:

- exchange any two rows
- multiply any row by a non-zero constant
- add any multiple of a row to another row

Gaussian-Jordan method

Use elementary row operations to transform matrix A of the augmented matrix $(A|b)$ to the diagonal form.

Solving systems of linear equations:

- apply Gaussian elementary operation to reduce the augmented matrix to the echelon form
- obtain x by back substitution

Gauss elimination method for augmented matrix $(A|b)$:

Transformation of $(A|b)$ to the upper triangular form:

For $s = 1, 2, 3, \ldots, n-1$:

for $r = s+1, \ldots, n$:

for $k = s+1, \ldots, n$:

$$A_{rk} = A_{rk} - A_{sk} \cdot A_{rs}/A_{ss}$$

$$b_r = b_r - b_s \cdot A_{rs}/A_{ss}$$

$$A_{rs} = 0$$

Partial and complete **pivoting** - to avoid lost of information.

Example: Notice the difference between results obtained by GEA without pivoting and GEA with pivoting:

$$\begin{aligned} 10^{-8}x + 10^{9}y &= 10^{9} \\ 10^{2}x + 10^{2}y &= 0. \end{aligned}$$

Number of floating point operations.

Note that

$$n(n-1) + (n-1)(n-2) + \ldots + 3 \cdot 2 + 2 \cdot 1 = \frac{1}{3}n(n-1)^2,$$

and

$$1 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

What is the number of multiplications and divisions needed for GEA?

FLOPS or flops - an acronym for FLoating-point Operations Per Second:

You may run the GEA for many large matrices and find out what is FLOPS of your computer.

Back substitution.

Computation of the solution x if A is in the upper triangular form.

For $r = n, \ldots, 1$:

$x_r = b_r$

for $s = n, \ldots, r + 1$:

$x_r = x_r - A_{rs} \cdot x_s$

$x_r = x_r / A_{rr}$

Let A be in the form of the product of L and U, $A = LU$, where L is lower triangular and U is upper triangular. Then x from $Ax = b$ is easy to obtain, because

$$Ax = LUx = b$$

can be solved in two easy steps

$$Ly = b \qquad \text{and} \qquad Ux = y.$$

LU decomposition (input A; output L and U)

For $s = 1, \ldots, n$:

for $r = 1, \ldots, s$:

$$U_{rs} = A_{rs} - \sum_{k=1}^{r-1} L_{rk} U_{ks}$$

for $r = r + 1, \ldots, n$:

$$L_{rs} = (A_{rs} - \sum_{k=1}^{s-1} L_{rk} U_{ks}) / U_{ss}.$$

If A is symmetric we can get $A = LL^T$ and again, $Ax = b$ can be obtained from

$$Ly = b \qquad \text{and} \qquad L^T x = y.$$

Choleski decomposition. (input A; output L)

For $r = 1, \ldots, n$:

$$L_{rr} := \sqrt{A_{rr} - \sum_{s=1}^{r-1} L_{rs}^2}$$

for $k = r + 1, \ldots, n$:

$$L_{kr} := \frac{1}{L_{rr}} \left(A_{kr} - \sum_{s=1}^{r-1} L_{rs} L_{ks} \right)$$

No pivots are needed if A is positive definite.

Exercises and programming:

Exercises.

1. Write a script for an exchange of two rows/columns of A.
2. Write a script for finding the position of the largest/smallest element of A.
3. Write a script for matrix multiplication AB and compare with the Matlab command $A * B$.
4. Write a script for Gaussian elimination of A.
5. Write a script for LU-decomposition of A.
6. Write a script which solves $Ax = b$.
7. Solve by hand and using your script

$$\begin{aligned} 10^{-8}x + 10^{9}y &= 10^{10} \\ x + 10^{-2}y &= 10. \end{aligned}$$

Two final tasks:

1. Solve the 5×5 equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \\ 16 \\ 25 \end{pmatrix}.$$

2. Solve the $n \times n$ equation

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}.$$

a) What is x_m, for $m = \frac{n}{2}$ (n even) or $m = \frac{n+1}{2}$ (n odd)?

b) What is x_1 for $n = 10$?